



TITLE:

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(Some inequalities concerned with
the geometric function theory)

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CITATION:

Miyakoda, Tsuyako. On N-Fractional Calculus of the Function $((z-b)^2-c)^{\frac{1}{3}}$ (Some inequalities concerned with the geometric function theory). 数理解析研究所講究録 2014, 1878: 17-29

ISSUE DATE:

2014-04

URL:

<http://hdl.handle.net/2433/195605>

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On N-Fractional Calculus of the Function $((z - b)^2 - c)^{\frac{1}{3}}$

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Abstract

We discuss the N-fractional calculus of $f(z) = ((z - b)^2 - c)^{\frac{1}{3}}$. In order to do fractional calculus of $((z - b)^2 - c)^{\frac{1}{3}}$, we consider four type's factorization of the equation and calculate

1. $(f)_\gamma = \left(((z - b)^2 - c)^{\frac{1}{3}} \right)_\gamma$
2. $(f)_\gamma = \left(((z - b)^2 - c)^{-\frac{2}{3}} ((z - b)^2 - c) \right)_\gamma$
3. $(f)_\gamma = \left(((z - b)^2 - c)^{-\frac{5}{3}} ((z - b)^2 - c)^2 \right)_\gamma$
4. $(f)_\gamma = \left((((z - b)^2 - c)^2 - c)^{\frac{1}{3}} \right)_{\gamma-1}$

We have four representations of fractional calculus. And then we show that these four different forms of N-fractional calculus are consistent in special case. And some identities are reported.

1 Introduction

We adopt the following definition of the fractional calculus.

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = C(f)_\nu \\ &= \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{\nu+1}} \quad (\nu \notin \mathbb{Z}^-), \end{aligned} \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C_-, \quad 0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \text{ Gamma function,}$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}^-), \quad (\text{Refer to [1]}) \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation \circ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\alpha (N^\beta f) \quad (\alpha, \beta \in R), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [4]

(III)

In some previous papers, the following result are known as elementary properties.

Lemma. We have [1]

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

Moreover in the previous works we refer to the next theorem [6].

Theorem D. We have

(i)

$$\begin{aligned} (((z-b)^\beta - c)^\alpha)_\gamma &= e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta} \right)^k \\ &\quad (|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}| < \infty), \end{aligned} \tag{9}$$

and

(ii)

$$\begin{aligned} (((z-b)^\beta - c)^\alpha)_n &= (-1)^n (z-b)^{\alpha\beta-n} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta} \right)^k \\ &\quad (n \in \mathbb{Z}_0^+, \quad |\frac{c}{(z-b)^\beta}| < 1), \end{aligned} \tag{10}$$

where

$$[\lambda]_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \Gamma(\lambda + k)/\Gamma(\lambda) \quad \text{with } [\lambda]_0 = 1,$$

(Pochhammer's Notation).

2 N-Fractional Calculus of the Functions $f(z) = ((z - b)^2 - c)^{\frac{1}{3}}$

In order to have a representation of N-fractional calculus with γ -order, we directly apply the theorem to the function at the beginning.

Theorem 1. Let

$$f = f(z) = ((z - b)^2 - c)^{\frac{1}{3}} \quad \left(((z - b)^2 - c)^{\frac{1}{3}} \neq 0 \right) \quad (1)$$

we have

$$(f)_{\gamma} = e^{-i\pi\gamma} (z - b)^{-\frac{2}{3}-\gamma} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k! \Gamma(2k - \frac{2}{3})} \left(\frac{c}{(z - b)^2} \right)^k \quad (2)$$

Proof. According to Theorem D, we have the equation (1) directly.

Secondly, we consider the function as a product of two functions like as

$$f(z) = ((z - b)^2 - c)^{-\frac{2}{3}} \cdot ((z - b)^2 - c)$$

and we have the new representation for $(f)_{\gamma}$ as follows.

Theorem 2. We set $f = f(z)$, and S, K, J as follows,

$$S = S(z) = \frac{c}{(z - b)^2}, \quad (|S| < 1) \quad (3)$$

$$K(k, \gamma, m) = \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - m)}{k! \Gamma(2k + \frac{4}{3})} S^k, \quad (4)$$

$$J(\gamma, m) = \sum_{k=0}^{\infty} K(k, \gamma, m). \quad (5)$$

We have

$$(f)_{\gamma} = e^{-i\pi\gamma} (z - b)^{-\frac{4}{3}-\gamma+2} \{ (1 - S)J(\gamma, 0) - 2\gamma J(\gamma, 1) + \gamma(\gamma - 1)J(\gamma, 2) \} \quad (6)$$

Proof. According to Lemma (iv), we have

$$(f)_\gamma = \left(((z-b)^2 - c)^{-\frac{2}{3}} \cdot ((z-b)^2 - c) \right)_\gamma \quad (7)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma+1-k)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-k} \cdot ((z-b)^2 - c)_k \quad (8)$$

and applying Theorem D.(i) to

$$\left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-k}, \quad (9)$$

we obtain

$$\begin{aligned} (f)_\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_\gamma ((z-b)^2 - c)_0 \\ &\quad + \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-1} (2(z-b)) \\ &\quad + \frac{\Gamma(\gamma+1)}{2! \Gamma(\gamma-1)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-2} \cdot 2 \\ &= \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_\gamma ((z-b)^2 - c) + 2\gamma \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-1} \cdot (z-b) \\ &\quad + 2\gamma(\gamma-1) \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-2} \\ &= e^{-i\pi\gamma} (z-b)^{-\frac{4}{3}-\gamma} ((z-b)^2 - c) \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma)}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k \\ &\quad + 2\gamma(z-b) e^{-i\pi(\gamma-1)} (z-b)^{-\frac{4}{3}-\gamma+2} \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - 1)}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k \\ &\quad + 2\gamma(\gamma-1) e^{-i\pi(\gamma-2)} (z-b)^{-\frac{4}{3}-\gamma+2} \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - 2)}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k \end{aligned} \quad (10)$$

Then we have the representation

$$(f(z))_\gamma = e^{-i\pi\gamma} (z-b)^{-\frac{4}{3}-\gamma+2} \{ (1-S)J(\gamma, 0) - 2\gamma J(\gamma, 1) + 2\gamma(\gamma-1)J(\gamma, 2) \}. \quad (11)$$

This is the same one as the equation (6).

Next, we consider the function as another product form like as

$$f(z) = ((z-b)^2 - c)^{-\frac{5}{3}} \cdot ((z-b)^2 - c)^2$$

and we have the new representation for $(f)_\gamma$ as follows.

Theorem 3. We set $f = f(z)$, and S, H, G as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \quad (12)$$

$$H(k, \gamma, m) = \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma - m)}{k! \Gamma(2k + \frac{10}{3})} S^k, \quad (13)$$

$$G(\gamma, m) = \sum_{k=0}^{\infty} H(k, \gamma, m). \quad (14)$$

We have

$$(f)_\gamma = e^{-i\pi\gamma} (z-b)^{-\frac{10}{3}-\gamma+4} \{ (1-S)^2 G(\gamma, 0) - 4\gamma(1-S)G(\gamma, 1) + 6\gamma(\gamma-1)(1-\frac{1}{3}S)G(\gamma, 2) \\ - 4\gamma(\gamma-1)(\gamma-2)G(\gamma, 3) + \gamma(\gamma-1)(\gamma-2)(\gamma-3)G(\gamma, 4) \} \quad (15)$$

Proof. According to Lemma (iv), we have

$$(f)_\gamma = \left(((z-b)^2 - c)^{-\frac{5}{3}} \cdot (((z-b)^2 - c)^2) \right)_\gamma \quad (16)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k! \Gamma(\gamma+1-k)} (((z-b)^2 - c)^{-\frac{5}{3}})_{\gamma-k} \cdot (((z-b)^2 - c)^2)_k \quad (17)$$

and applying Theorem D.(i) to

$$(((z-b)^2 - c)^{-\frac{5}{3}})_{\gamma-k}, \quad (18)$$

we obtain

$$(f)_\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)} \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_\gamma \left(((z-b)^2 - c)^2 \right)_0 \\ + \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)} \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-1} (4((z-b)^2 - c)(z-b)) \\ + \frac{\Gamma(\gamma+1)}{2! \Gamma(\gamma-1)} \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-2} \cdot (12(z-b)^2 - 4c) \\ + \frac{\Gamma(\gamma+1)}{3! \Gamma(\gamma-2)} \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-3} \cdot (24(z-b)) \\ + \frac{\Gamma(\gamma+1)}{4! \Gamma(\gamma-3)} \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-4} \cdot 24$$

$$\begin{aligned}
&= \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_\gamma \left(((z-b)^2 - c)^2 + 4\gamma \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-1} \cdot ((z-b)^3 - c(z-b)) \right. \\
&\quad + \gamma(\gamma-1) \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-2} (6(z-b)^2 - 2c) \\
&\quad + 4\gamma(\gamma-1)(\gamma-2) \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-3} (z-b) \\
&\quad \left. + \gamma(\gamma-1)(\gamma-2)(\gamma-3) \left(((z-b)^2 - c)^{-\frac{5}{3}} \right)_{\gamma-4} \right) \\
&= e^{-i\pi\gamma} (z-b)^{-\frac{10}{3}-\gamma+4} ((z-b)^2 - c) \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma)}{k! \Gamma(2k + \frac{10}{3})} \left(\frac{c}{(z-b)^2} \right)^k \left(1 - \frac{c}{(z-b)^2} \right)^2 \\
&\quad + 4\gamma(z-b) e^{-i\pi(\gamma-1)} (z-b)^{-\frac{10}{3}-\gamma+4} \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma - 1)}{k! \Gamma(2k + \frac{10}{3})} \left(\frac{c}{(z-b)^2} \right)^k \left(1 - \frac{c}{(z-b)^2} \right) \\
&\quad + 6\gamma(\gamma-1) e^{-i\pi(\gamma-2)} (z-b)^{-\frac{10}{3}-\gamma+4} \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma - 2)}{k! \Gamma(2k + \frac{10}{3})} \left(\frac{c}{(z-b)^2} \right)^k \left(1 - \frac{1}{3} \frac{c}{(z-b)^2} \right) \\
&\quad + 4\gamma(\gamma-1)(\gamma-2) e^{-i\pi(\gamma-3)} (z-b)^{-\frac{10}{3}-\gamma+4} \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma - 3)}{k! \Gamma(2k + \frac{10}{3})} \left(\frac{c}{(z-b)^2} \right)^k \\
&\quad + \gamma(\gamma-1)(\gamma-2)(\gamma-3) e^{-i\pi(\gamma-4)} (z-b)^{-\frac{10}{3}-\gamma+4} \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma - 4)}{k! \Gamma(2k + \frac{10}{3})} \left(\frac{c}{(z-b)^2} \right)^k
\end{aligned} \tag{19}$$

Then we have the representation

$$\begin{aligned}
(f(z))_\gamma &= e^{-i\pi\gamma} (z-b)^{-\frac{10}{3}-\gamma+4} \{ (1-S)^2 G(\gamma, 0) - 4\gamma(1-S)G(\gamma, 1) + 6\gamma(\gamma-1)(1-\frac{1}{3}S)G(\gamma, 2) \\
&\quad - 4\gamma(\gamma-1)(\gamma-2)G(\gamma, 3) + \gamma(\gamma-1)(\gamma-2)(\gamma-3)G(\gamma, 4) \}.
\end{aligned} \tag{20}$$

This is the same one as the equation (15).

Next, we choose another process of the fractional calculus which is divided into two stages as like as

$$(f(z))_\gamma = ((f(z))_1)_{\gamma-1}. \tag{21}$$

We have an another result.

Theorem 4. We set $f = f(z)$, and S, R, W as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \tag{22}$$

$$R(k, \gamma, m) = \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - m)}{k! \Gamma(2k + \frac{4}{3})} S^k, \tag{23}$$

$$W(\gamma, m) = \sum_{k=0}^{\infty} R(k, \gamma, m). \quad (24)$$

Then we have

$$(f)_{\gamma} = \frac{2}{3} e^{-i\pi\gamma} (z-b)^{-\frac{4}{3}-\gamma+2} \{-W(\gamma, 1) - (\gamma-1)W(\gamma, 2)\}. \quad (25)$$

Proof. We have

$$\begin{aligned} \left(((z-b)^2 - c)^{\frac{1}{3}} \right)_1 &= \frac{1}{3} ((z-b)^2 - c)^{-\frac{2}{3}} \cdot 2(z-b) \\ &= \frac{2}{3} ((z-b)^2 - c)^{-\frac{2}{3}} (z-b) \end{aligned} \quad (26)$$

Then

$$\begin{aligned} \left((((z-b)^2 - c)^{\frac{1}{3}})_1 \right)_{\gamma-1} &= \frac{2}{3} \left(((z-b)^2 - c)^{-\frac{2}{3}} (z-b) \right)_{\gamma-1} \\ &= \frac{2}{3} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma)}{k! \Gamma(\gamma-k)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-1-k} (z-b)_k \\ &= \frac{2}{3} \left\{ \frac{\Gamma(\gamma)}{\Gamma(\gamma)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-1} (z-b) + \frac{\Gamma(\gamma)}{\Gamma(\gamma-1)} \left(((z-b)^2 - c)^{-\frac{2}{3}} \right)_{\gamma-1-1} \right\} \\ &= \frac{2}{3} \{ e^{-i\pi(\gamma-1)} (z-b)^{-\frac{4}{3}-\gamma+2} \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - 1)}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k \\ &\quad + (\gamma-1) e^{-i\pi(\gamma-2)} (z-b)^{-\frac{4}{3}-\gamma+2} \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - 2)}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k \} \end{aligned} \quad (27)$$

And we put

$$R(k, \gamma, m) = \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + \gamma - m)}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k,$$

$$W(\gamma, m) = \sum_{k=0}^{\infty} R(k, \gamma, m).$$

So we have

$$(f(z))_{\gamma} = \frac{2}{3} e^{-i\pi\gamma} (z-b)^{-\frac{4}{3}-\gamma+2} \{-W(\gamma, 1) + (\gamma-1)W(\gamma, 2)\}, \quad (\gamma \notin \mathbb{Z}^-). \quad (28)$$

We have the equation (25) from above equation directly.

3 Identities

We have four kinds of representation on N-fractional calculus of the function $f(z) = ((z-b)^2 - c)^{-\frac{1}{3}}$ like as Theorem 1, 2, 3 and 4. Accordingly we have the following identities with using J and G and W and L given in §2.

Theorem 5. We have

(i)

$$\sum_{k=0}^{\infty} \frac{[\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k! \Gamma(2k - \frac{2}{3})} S^k = (1-S)J(\gamma, 0) - 2\gamma J(\gamma, 1) + 2\gamma(\gamma-1)J(\gamma, 2), \quad (\gamma \notin Z^-) \quad (1)$$

and

(ii)

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{[\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k! \Gamma(2k - \frac{2}{3})} S^k &= (1-S)^2 G(\gamma, 0) - 4\gamma(1-S)G(\gamma, 1) \\ &\quad + 6\gamma(\gamma-1)(1-\frac{1}{3}S)G(\gamma, 2) - 4\gamma(\gamma-1)(\gamma-2)G(\gamma, 3) \\ &\quad + \gamma(\gamma-1)(\gamma-2)(\gamma-3)G(\gamma, 4), \quad (\gamma \notin Z^-) \end{aligned} \quad (2)$$

(iii)

$$\sum_{k=0}^{\infty} \frac{[\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k! \Gamma(2k - \frac{2}{3})} S^k = \frac{2}{3} \{-L(\gamma, 1) + (\gamma-1)L(\gamma, 2)\}. \quad (\gamma \notin Z^-) \quad (3)$$

Proof. From Theorems 2 and 3 and 4 we can obtain above equations directly.

4 A Special Case

In order to make sure of the formulations of Theorem 1, 2, 3 and 4, we consider the case of the integer $\gamma = 1$.

From Theorem 1, in case of $\gamma = 1$ the equation becomes

$$(f)_1 = e^{-i\pi}(z-b)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + 1)}{k! \Gamma(2k - \frac{2}{3})} S^k$$

$$\begin{aligned}
&= e^{-i\pi}(z-b)^{-\frac{1}{3}} \left\{ 2 \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k k}{k!} S^k - \frac{2}{3} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k k}{k!} S^k \right\} \\
&= e^{-i\pi}(z-b)^{-\frac{1}{3}} \left\{ 2S(-\frac{1}{3}) \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k}{k!} S^k - \frac{2}{3} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k k}{k!} S^k \right\} \\
&= e^{-i\pi}(z-b)^{-\frac{1}{3}} \left\{ 2S(-\frac{1}{3})(1-S)^{-\frac{2}{3}} - \frac{2}{3}(1-S)^{\frac{1}{3}} \right\} \\
&= (-1)(z-b)^{-\frac{1}{3}} \left(-\frac{2}{3}\right)(1-S)^{-\frac{2}{3}} \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}
\end{aligned} \tag{1}$$

When $\gamma = 1$, from Theorem 2 , we have

$$\begin{aligned}
(f)_1 &= e^{-i\pi}(z-b)^{-\frac{1}{3}} \{(1-S)J(1,0) - 2J(1,1)\} \\
&= (-1)(z-b)^{-\frac{1}{3}} \left\{ (1-S) \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3} + 1)}{k! \Gamma(2k + \frac{4}{3})} S^k \right. \\
&\quad \left. - 2 \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3})}{k! \Gamma(2k + \frac{4}{3})} S^k \right\}
\end{aligned} \tag{2}$$

And we notice following relations,

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda} \tag{3}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{[\lambda]_k k}{k!} T^k &= \sum_{k=0}^{\infty} \frac{[\lambda]_k}{(k-1)!} T^k = \sum_{k=0}^{\infty} \frac{[\lambda]_{k+1}}{k!} T^{k+1} \\
&= \lambda T \sum_{k=0}^{\infty} \frac{[\lambda+1]_k}{k!} T^k = \lambda T (1-T)^{-1-\lambda}
\end{aligned} \tag{4}$$

$$[\lambda]_{k+1} = \frac{\Gamma(\lambda+1+k)}{\Gamma(\lambda)} = \lambda[\lambda+1]_k \tag{5}$$

Then, we have the following relations with applying to the above euations.

$$\begin{aligned}
(f)_1 &= -(z-b)^{-\frac{1}{3}} \left\{ 2S(1-S) \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_{k+1}}{k!} S^k + \frac{4}{3}(1-S) \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k}{k!} S^k \right\} - 2 \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k}{k!} S^k \\
&= -(z-b)^{-\frac{1}{3}} \left\{ 2(1-S)S \left(\frac{2}{3}\right) \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k}{k!} S^k + \frac{4}{3}(1-S)(1-S)^{-\frac{2}{3}} - 2(1-S)^{-\frac{2}{3}} \right\}
\end{aligned}$$

$$\begin{aligned}
&= -(z-b)^{-\frac{1}{3}} \left\{ \frac{4}{3}(1-S)S(1-S)^{-\frac{5}{3}} + \frac{4}{3}(1-S)(1-S)^{-\frac{2}{3}} - 2(1-S)^{-\frac{2}{3}} \right\} \\
&= -(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}} \left(\frac{4}{3}S + \frac{4}{3} - \frac{4}{3}S - 2 \right) \\
&= \frac{2}{3}(z-b)^{\frac{1}{3}}(1-S)^{-\frac{2}{3}}.
\end{aligned} \tag{6}$$

And from Theorem 3, we have

$$\begin{aligned}
(f)_1 &= e^{-i\pi}(z-b)^{-\frac{1}{3}} \{(1-S)^2 G(1,0) - 4(1-S)G(1,1)\} \\
&= e^{-i\pi}(z-b)^{-\frac{1}{3}}(1-S) \left\{ (1-S) \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + 1)}{k! \Gamma(2k + \frac{10}{3})} S^k \right. \\
&\quad \left. - 4 \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3})}{k! \Gamma(2k + \frac{10}{3})} S^k \right\} \\
&= -(z-b)^{-\frac{1}{3}}(1-S) \left\{ 2(1-S)S \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_{k+1}}{k!} S^k \right. \\
&\quad \left. + \frac{10}{3}(1-S) \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k}{k!} S^k - 4 \sum_{k=0}^{\infty} \frac{[\frac{5}{3}]_k}{k!} S^k \right\} \\
&= -(z-b)^{-\frac{1}{3}}(1-S) \left\{ \frac{10}{3}S(1-S)^{\frac{5}{3}} + \frac{10}{3}(1-S)^{-\frac{2}{3}} - 4(1-S)^{-\frac{5}{3}} \right\} \\
&= -(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}} \left(\frac{10}{3}S - 4 + \frac{10}{3} - \frac{10}{3}S \right) \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}.
\end{aligned} \tag{7}$$

Next, from Theorem 4 we have

$$\begin{aligned}
(f)_1 &= \frac{2}{3}(-1)(z-b)^{-\frac{4}{3}+1} \{-L(1,1)\} \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}} L(1,1) \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k \Gamma(2k + \frac{4}{3})}{k! \Gamma(2k + \frac{4}{3})} \left(\frac{c}{(z-b)^2} \right)^k \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k}{k!} \left(\frac{c}{(z-b)^2} \right)^k \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}} \left(1 - \frac{c}{(z-b)^2} \right)^{-\frac{2}{3}} \\
&= \frac{2}{3}(z-b) \left((z-b)^2 - c \right)^{-\frac{2}{3}}
\end{aligned} \tag{8}$$

Therefore we have the same results from four different forms of N-fractional calculus for the function $((z-b)^2 - c)^{\frac{1}{3}}$.

Now these results are consistent with the one of the classical calculus of

$$\frac{d}{dz}((z-b)^2 - c)^{\frac{1}{3}}. \quad (9)$$

Here we confirm again the result for Theorem 1.

When $\gamma = 1$, from Theorem 1.(2), we have

$$\begin{aligned} \left(((z-b)^2 - c)^{\frac{1}{3}} \right)_1 &= -(z-b)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k! \Gamma(2k - \frac{2}{3})} S^k \\ &= -(z-b)^{-\frac{1}{3}} \left\{ 2 \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k k}{k!} S^k - \frac{2}{3} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k}{k!} S^k \right\} \\ &= -(z-b)^{-\frac{1}{3}} \left\{ 2S(-\frac{1}{3}) \sum_{k=0}^{\infty} \frac{[\frac{2}{3}]_k}{k!} S^k - \frac{2}{3} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k}{k!} S^k \right\} \\ &= -(z-b)^{-\frac{1}{3}} \left\{ -\frac{2}{3} S(1-S)^{-\frac{2}{3}} - \frac{2}{3} (1-S)^{\frac{1}{3}} \right\} \\ &= -(z-b)^{-\frac{1}{3}} (1-S)^{-\frac{2}{3}} \left\{ -\frac{2}{3} S - \frac{2}{3} (1-S) \right\} \\ &= -(z-b)^{-\frac{1}{3}} \left(-\frac{2}{3} \right) (1-S)^{-\frac{2}{3}} \\ &= \frac{2}{3} (z-b)^{-\frac{1}{3}} (1-S)^{-\frac{2}{3}} \end{aligned} \quad (10)$$

We have

$$\begin{aligned} \frac{2}{3} (z-b)^{-\frac{1}{3}} (1-S)^{-\frac{2}{3}} &= \frac{2}{3} (z-b)^{-\frac{1}{3}} \left(\frac{(z-b)^2 - c}{(z-b)^2} \right)^{-\frac{2}{3}} \\ &= \frac{2}{3} (z-b) ((z-b)^2 - c)^{-\frac{2}{3}}. \end{aligned} \quad (11)$$

This result also coincides with the one obtained by the classical calculus.

So we conclude that according to the definition of fractional differentiation, we have three forms for γ -th differintegrate of the function $((z-b)^2 - c)^{\frac{1}{3}}$ by Theorems 1, 2, 3 and 4.

We made sure that they have the same results as the classical result when the differential order is in the case of $\gamma = 1$.

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